Constructing Default Boundaries

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June, 2005

Question

To construct a framework of credit risk which is flexible enough to deal with both credit risk and equity risk (and interest rate risk preferably)

The Structural Approach to Credit Risk Valuation

Asset Pricing

- Black and Scholes (1973) and Merton (1974)
  - The contingent claim approach
- Black and Cox (1976), Longstaff and Schwartz (1995), Leland and Toft (1996), Collin-Dufresne and Goldstein (2001)
  - Exogenous vs endogenous default boundaries
- Commercial applications
  - Moody's KMV, CreditGrades, and many others

Assessing Default Probability

- Empirical Studies: Huang and Huang (2004), and Leland (2004) etc
- Maths Finance: Presented below

Economical meaningful but technically very difficult

The Reduced Form Approach to Credit Risk Valuation

- Model default intensity process directly
  - Jarrow and Turnbull (1995), Jarrow, Lando and Turnbull (1997), Jarrow and Fan (2001), and Duffie and Singleton (1997, 1999),Duffie, Pan and Singleton (2000)
- **Where is Capital Structure?**
- **Structural Versus Reduced Form Models**
  - Imperfect/Partial information
  - Duffie and Lando (2001), Cetin, Jarrow, Protter, and Yildirim (2004).

An ultimate research goal $\Rightarrow$ convergence of both structural and reduced-form approaches?

An Alternative Approach

- To construct a structural model to reflect credit risk information
- To link structural approach and reduced form approach
- To address equity risk from credit risk

Inverse of the First Passage Time Problem

▶ Maths Finance Problems

- (Ω, F, P), and a standard Brownian motion $\{W(t)\}$ with $W(0) = 0$
- A continuous function $\{b(t)\}$ such that $b(0) > 0$.
- The first passage time and probability function:

$$\tau_b = inf\{t : W(t) \geq b(t)\}, P_b(t) = P(\tau_b \leq t)$$

- **First Passage Time Probability Problem**: To derive $P_b(.)$ by giving $b(.)$.
- **Inverse of the First Passage Time Problem:** Given a probability function $P(.)$ to find a curve $b(.)$ such that $P(t) = P_b(t)$?

▶ General Consideration on Limit at Zero

- The density function $f(t)$ of τ_b: $f(t) = P'(t)$.
- The hazard rate $h(t) = \frac{P'(t)}{1-P(t)}$
- $f(0+) = h(0+) = 0$
- $f(0+)$ and $h(0+)$ might be positive for the case that $b(0) = 0$ or $b(.)$ is not continuous.

Firm Value Model

- $V(t) = F(W(t), t)$ where $F(x, t)$ is positive, smooth enough, and $F_x(x, t) > 0$
- Diffusion process $dV(t) = \mu(V, t)dt + \sigma(V, t)dW(t)$
- A curve for the firm value corresponding uniquely to an curve for the Brownian motion. $B(t) = F(b(t), t)$. Both $\{B(.)\}$ and $\{b(.)\}$ are determined each other
- A default threshold $B(0)$ of the firm value, which implies a starting point $b(0)$ for the corresponding $W(t)$
- **Lognormal example:** $V(t) = V(0)exp\{(\mu - \frac{1}{2}\sigma^2)t - \sigma W(t)\}$. Then

$$b(t) = \frac{log(V(0)/B(t))}{\sigma} + \frac{\mu - \frac{1}{2}\sigma^2}{\sigma}t$$

and

$$B(t) = V(0)exp\{(\mu - \frac{1}{2}\sigma^2)t - \sigma b(t)\}$$

and

$$b(0) = \frac{log(V(0)/B(0))}{\sigma}, B(0) = V(0)exp\{-\sigma b(0)\}$$

- debt-per-share implies $V(0)/B(0)$. Therefore, the starting point of the curve $b(.)$ depends on debt-per-share

Constructing Default Boundary

▶ Previous Construction of the Default Boundary

- A special function form $B(t) = exp\{\alpha + \beta t\}$ (the first passage model)
- Assumption on the default boundary which essentially lead to special function form
- Analytical convenience

▶ An Implied Approach

- **Input**: Credit Risk Information (CDS spreads, or yield spreads, or default probabilities)
- **Output**: Implied default boundary to calibrate the given credit risk information
- Lead to the inverse problem

Practical Objective

Given a finite many of probability $\{P(0) = 0 < P(t_1) < P(t_2) < \ldots < P(t_n)\}$, and a fixed $b(0) > 0$, to find a continuous function $\{b(.)\}$ such that $P_b(t_i) = P(t_i)$ for each t_i.

▶ **We need to find a class of continuous functions such that**

- Easy to calculate the first passage time probability
- Easy to solve the inverse of the first passage time probability
- The class is larger enough
- The class includes the linear function class (the first passage model)

Implementation

- Piecewise Linear Continuous Function class

- Other classes

Given $0 = t_0 < t_1 < \ldots < t_n = T$, a piecewise continuous function $b(t)$ which is linear on each interval $[t_{j-1}, t_j], j = 1, 2, \ldots, n$ and $b(0) > 0$. Write $b_j = b(t_j), j = 1, 2, \ldots, n$. Then

$$P_b(t_k) = 1 - Eg(W(t_1), \ldots, W(t_k)), k = 1, \ldots, n \tag{1}$$

where

$$g(x_1, x_2, \ldots, x_k) = \prod_{j=1}^{k} 1_{\{x_j < b_j\}}\{1 - exp[-\frac{2(b_{j-1} - x_{j-1})(b_j - x_j)}{t_j - t_{j-1}}]\}$$

Example $n = 2$:

$$P_b(t_2) = 1 - \int\int_{\mathcal{A}} f(\zeta,\eta)n(\zeta)n(\eta)d\zeta d\eta$$

where

$$\mathcal{A} = \{(\zeta,\eta) : \sqrt{T_1}\zeta < b_1, \sqrt{T_1}\zeta + \sqrt{T_2 - T_1}\eta < b_2\}$$

and

$$\begin{aligned} f(\zeta,\eta) &= \{1 - exp(-\frac{2b_0(b_1 - \sqrt{T_1}\zeta)}{T_1})\} \\ &\quad \times\{1 - exp(-\frac{2(b_1 - \sqrt{T_1}\zeta)(b_2 - \sqrt{T_1}\zeta - \sqrt{T_2 - T_1}\eta)}{T_2 - T_1})\} \end{aligned}$$

Inverse Problem

Given $0 = t_0 < t_1 < \ldots < t_n = T$ **and numbers** $0 = A_0 < A_1 < \ldots < A_n \leq 1$**, and a given** $b(0) > 0$**, there exists a continuous function** $b_n(.)$ **such that** $P_{b_n}(t_i) = A_i, i = 1, \ldots, n$ **and** $b_n(0) = b(0)$**.**

t	$P(t)$	$b(t)$
$t = 0$	0	1.5
$t = 1$	0.05%	3.9956
$t = 2$	0.17%	4.6818
$t = 3$	0.35%	5.4637
$t = 4$	0.60%	6.4055

An Example

Panel A. Default Probabilities and Yield Spreads

T	1	2	3	4	5	6
Default Prob	2.96%	5.82%	8.61%	11.31%	13.93%	16.46%
Yield Spread	0.03	0.03	0.03	0.03	0.03	0.03

Panel B. Default Probabilities and Hazard Rate (piecewise constant)

T	1	2	3	4	5	6
Default Prob	0.64%	2.10%	3.81%	6.15%	8.12%	10.09%
Hazard Rate	0.0064	0.0210	0.0381	0.0615	0.0812	0.1009

Table 1: Default Probabilities, Yield Spreads and Hazard Rates

Data

Analysis

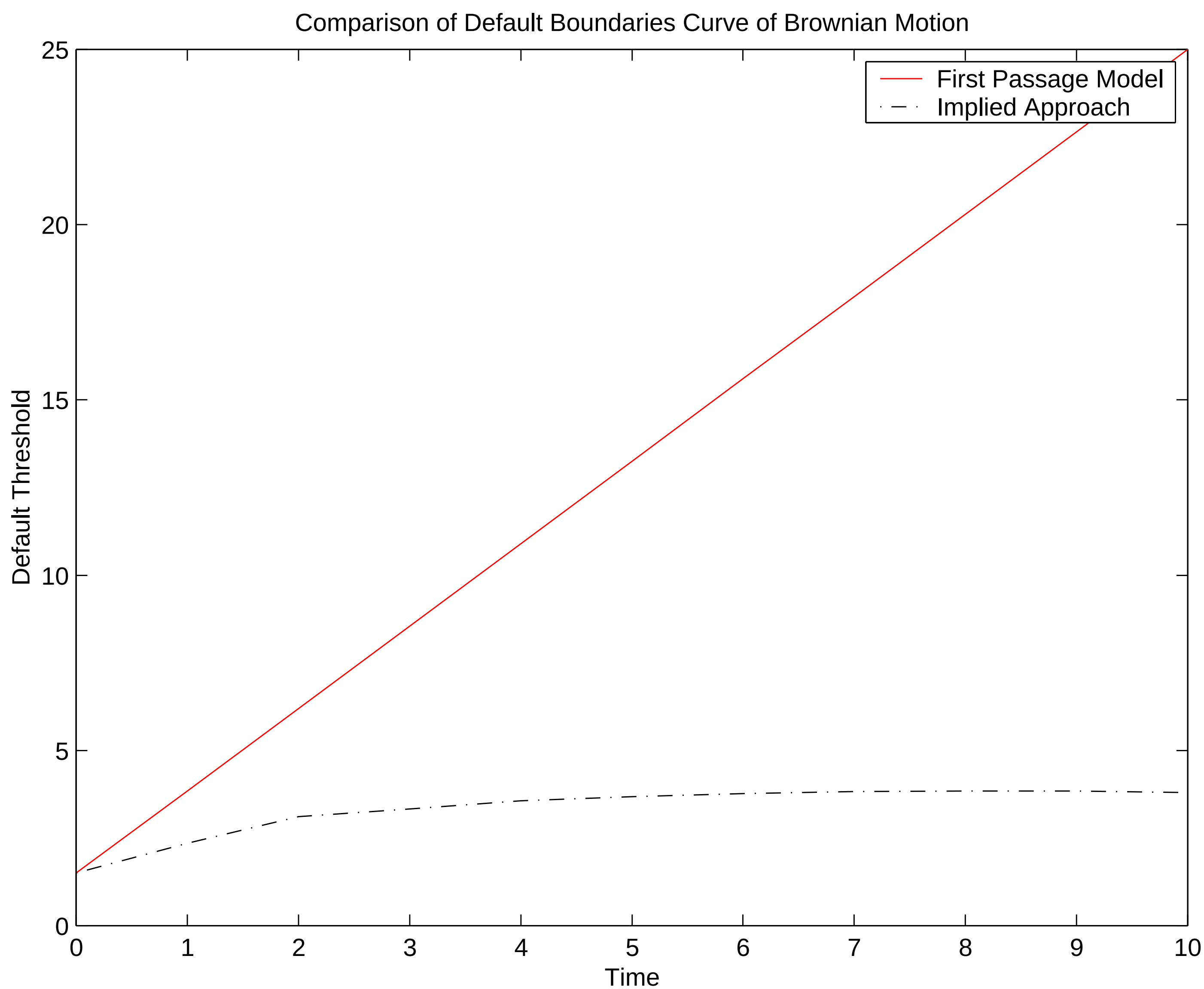


Figure 1: Constant yield spreads 300 bp

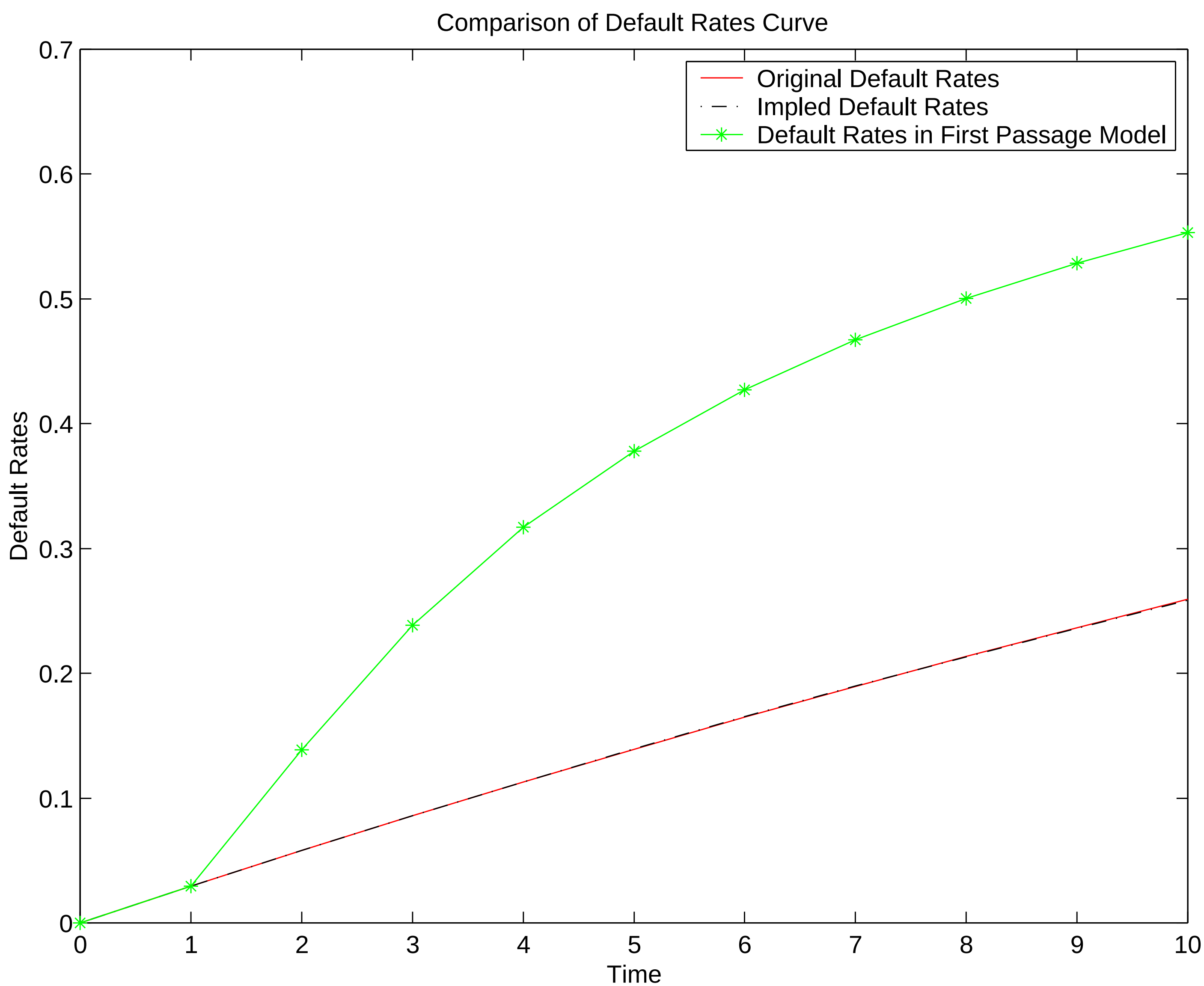
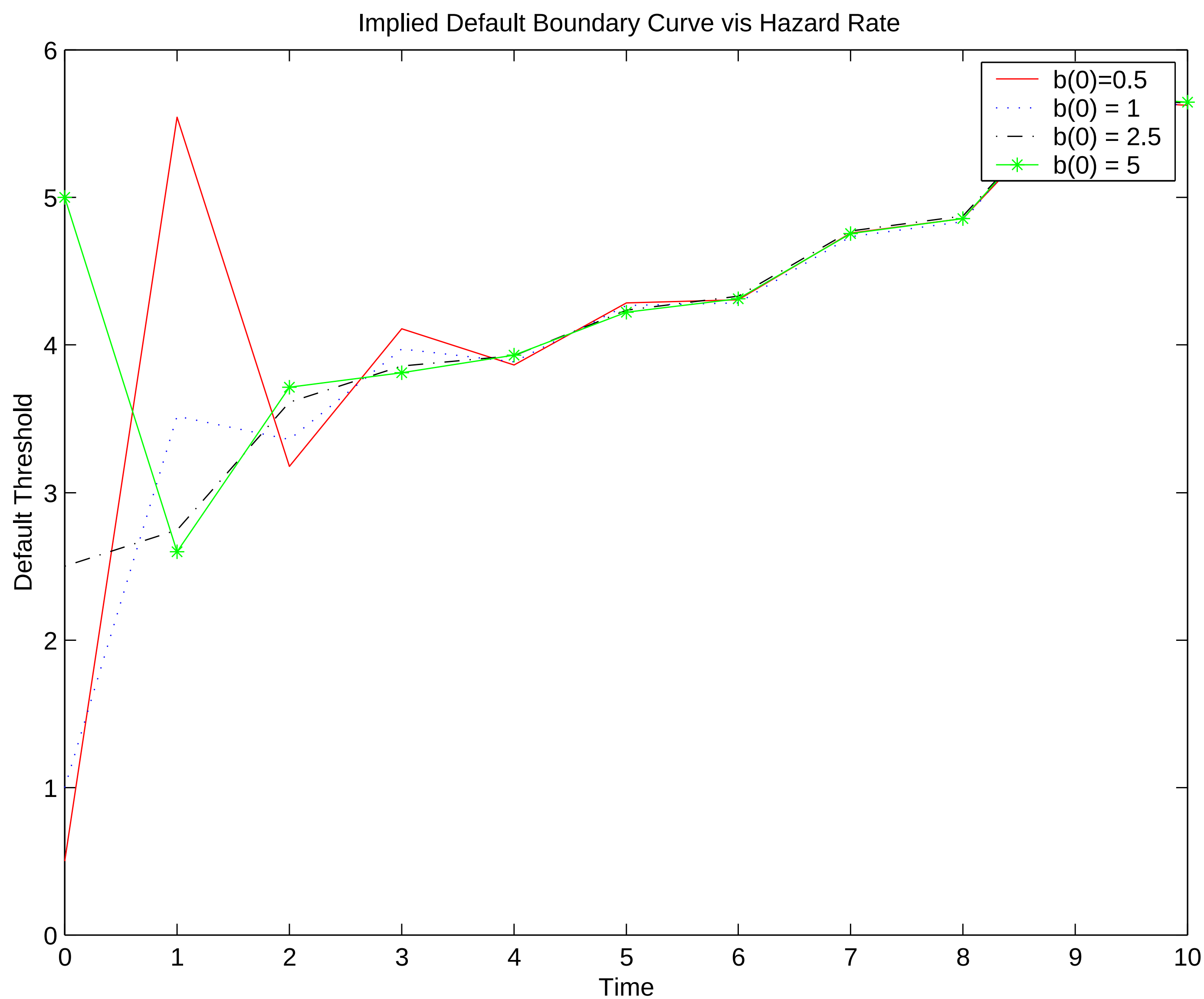
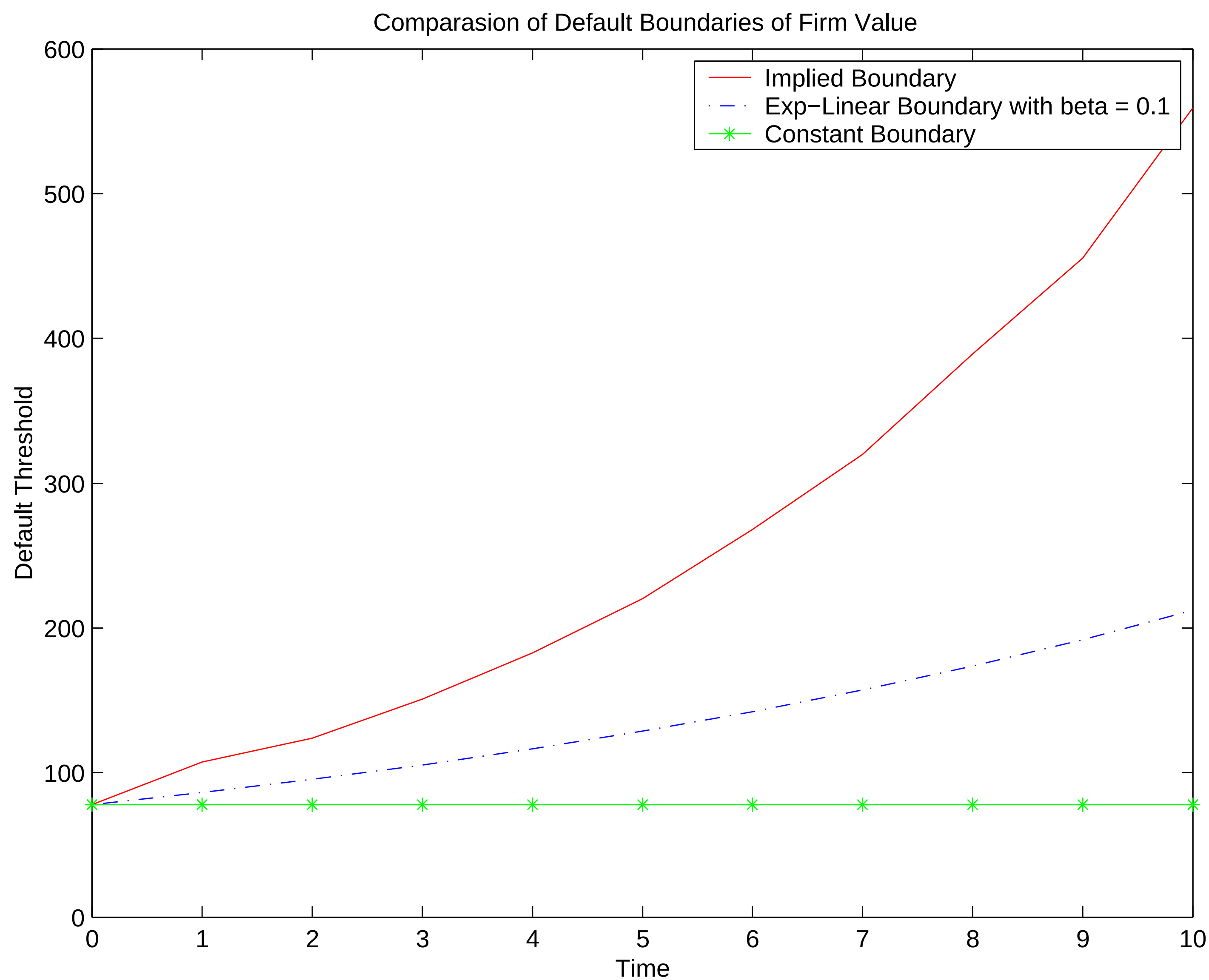


Figure 2: Constant yield spreads 300 bp


Implied Default Boundary Curve vis Hazard Rate
Default Threshold
Time
0
1
2
3
4
5
6
7
8
9
10
b(0)=0.5
b(0) = 1
b(0) = 2.5
b(0) = 5



$$\sigma = 5\%, r = 6\%, \delta = 5\%, \mu = 6.125\%, V_0 = 100, \alpha = 78, \beta = 0$$

Credit Risk from Reduced form Model

- **Inverse Problem: To match the continuous probability function for all future time**
- Impossible! But
- **Given a continuous probability function $\{P(t)\}$, and $b(0) > 0$, there exists a sequence of continuous functions $\{b_n(.)\}$, starting from $b_n(0) = b(0)$, such that**

$$\lim_{n \to \infty} P_{b_n}(t) = P(t), \forall t \in [0, T]$$

- **Given a default time τ with continuous probability function, and $b(0) > 0$, there exists a sequence of continuous functions $\{b_n(.)\}$, starting from $b_n(0) = b(0)$, such that $\tau_{b_n} \to \tau$ in distribution**

t	CDS Spread (bp)	Default Probability	Implied Default Boundary
$t=1$	150	3.61%	88.65
$t=2$	170	4.57%	77.68
$t=3$	200	6.26%	80.41
$t=4$	230	7.63%	76.77
$t=5$	250	7.96%	71.44
$t=6$	280	10.51%	77.68

Table 2: Recovery rate:40%, interest rate: 6%, firm value volatility: 6%, $\mu = 6.125\%, \delta = 5\%, V(0) = 100, B(0) = 78$.

A Hypothetical Example

E2C

- **Equity Market** + Assumptions on **Capital Structure** and assumptions on **equity versus firm value** $\Rightarrow$:
  - Default Probability (first passage model or modified)
  - CDS spreads
  - Goldman Sachs models, etc

C2E

- **Credit Market** + assumptions on **equity versus firm value** ⇒ equity option premium
- No assumption on the capital structure.
  - Too difficult to deal with a general given capital structure, in the structural approach.
  - Partial information?
- Equity price assumption: $S(t) = G(V(t), B(t))$, and approaches to zero when $V(t) \downarrow B(t)$. Example: $S(t) = V(t) - B(t)$ if $V(t) \geq B(t)$.
- Other issues
  - Calibration
  - Volatility Smile

An Example of Credit Protection

Assume that $S(t) = V(t) - B(t)$ when $V(t) \geq B(t)$. This assumption need to be modified in reality. Consider an equity put option with maturity 3 yr and strike K. In this framework, the equity put option is a "knock-out" barrier option written on the firm value with barrier $\{B(.)\}$, and the payoff at maturity is

$$[K - (V(3) - B(3))^{+}]^{+}$$

What is the credit protection of $ K in three years?

Strike	Equity Put	Credit Protection	Equity Implied Vol
10	0.4903	0.5254	77%
15	1.2614	0.7882	51%
20	2.6752	1.0509	38%
25	4.7936	1.3136	30%
30	7.6321	1.5763	23%

Table 3: Comparison of Credit Protections via Equity and Credit Market

Concluding Remarks

- New techniques of the first passage time probability literatures to solve the first passage time problem and its inverse problem, with applications to structural approach
- Alternative link between structural and redeuced form approach, and its potential to a uniform credit risk theory
- Alternative approach to capital structure arbitrage: C2E